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JOINS OF POLYHEDRA

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§1. INTRODUCTION

THIS paper is concerned with the structure of the set of compact polyhedra under the operation of join. It developed from the result, proved jointly with M. A. Armstrong for use in [1], that two polyhedra are PL homeomorphic if their suspensions are PL homeomorphic.

The main result, Theorems 3 and 4, is that any polyhedron can be expressed uniquely as the join of a maximal ball or sphere and other factors which are indecomposable as joins. This gives a decomposition of any polyhedron into indecomposable factors, which fails to be unique only in the decomposition of the maximal ball, which may be the repeated cone on a point, or the repeated suspension of a point, or some other mixture of cones and suspensions.

If we let T be a point, Σ be a pair of points and $*$ denote join, then $T * X$ and $\Sigma * X$ are respectively the cone on X and the suspension of X. We shall write $A = B$ to mean A is PL homeomorphic to B. Then the following cancellation law is a corollary to Theorems 3 and 4.

If $A * B = A * C$,

then either (i) $A = T * A'$, B and C are $T * X$ and $\Sigma * X$ for some polyhedra A' and X, or (ii) $B = C$.

A short section on self-homeomorphisms of polyhedra which are not cones or suspensions follows from the work on factorization.

§2. NOTATION AND DEFINITIONS

We shall use the terminology of Zeeman [2]. Our objects, *compact polyhedra*, are topological spaces with an equivalence class of PL related triangulation by finite complexes. We shall take every complex to contain the empty simplex of dimension -1.

If X and Y are PL homeomorphic, we shall write $X = Y$, or further $X \overset{h}{\cong} Y$ to mean that triangulations K and L of X and Y and a simplicial isomorphism h from K to L have been chosen. A simplex of X will then mean a simplex of the chosen triangulation K.

The *join* of two polyhedra X and Y will be written XY. There is a natural family of triangulations making it a polyhedron also. This family includes the triangulations by the complex $K * L$, called a *join triangulation*, whose simplexes are the joins of all possible pairs of simplexes, one from K and one from L, where K and L are any triangulating complexes of X and Y. The triangulation by $K * L$ contains K and L as disjoint subcomplexes triangulating the factors X and Y. We shall always take a subdivision of such a triangulation when choosing a triangulation for XY. A simplex in any triangulation of XY chosen thus, which does not lie in either of the factors X or Y is said to lie *between X and Y*.

Under the operation of join, the set of PL homeomorphism classes of compact polyhedra forms an abelian semigroup. We shall investigate the structure of this semigroup and show that it has very simple factorization properties. We shall therefore make the natural definitions of indecomposable and prime polyhedra.

An *indecomposable polyhedron* is not the join of two polyhedra. A *prime polyhedron* must divide one of the factors of a join if it divides the join.

Every polyhedron has a finite dimension, and because of the increase of dimension under join, every polyhedron can be written as the join of a finite number of indecomposables.

The polyhedron consisting of one point will be called T, and two points will be called Σ. This means that the PL sphere S^{r-1} is written as Σ^r, and the $(r-1)$-ball is T^r.

The link of a simplex α in the complex A will be written as $\mathrm{lk}(\alpha, A)$. This link will be considered as a polyhedron, rather than a subcomplex of A. When A is the largest complex under consideration we shall simply write $\mathrm{lk}\alpha$. Every point of a complex is contained in a principal simplex, whose link is empty. If we introduce the empty polyhedron, $\varnothing$, then it acts as the identity element in the semigroup, and we must make the obvious modification to the definition of indecomposable.

RESULTS ABOUT LINKS. *For proofs see* [2].

(lk 1) *The link of a point x in a polyhedron X can be well-defined, as a polyhedron, as the link of x in any triangulation of X having x as a vertex. For the technique of pseudo-radial projection will show that the links of x in a triangulation and in a subdivision are PL homeomorphic.*

(lk 2) *In a join triangulation of AB, the link of the simplex ab is* $\mathrm{lk}(a, A)\, \mathrm{lk}(b, B)$, *with the convention that if $a = \varnothing$ then* $\mathrm{lk}(a, A) = A$.

(lk 3) *If a point x lies in the interior of a simplex a in some triangulation of A, then* $\mathrm{lk}(x, A) = \Sigma^r\, \mathrm{lk}(a, A)$, *where $r = \dim a$.*

(lk 4) *Any point x in the polyhedron AB which does not lie in A or B must lie in the join of points a in A and b in B. Hence* $\mathrm{lk}\, x = \Sigma\, \mathrm{lk}(a, A)\, \mathrm{lk}(b, B)$.

§3. MAIN RESULTS

THEOREM 1. *Suspensions cancel. Suppose $\Sigma^r X = \Sigma_r Y$, where X and Y are not suspensions. Then $r = k$ and $X = Y$. Hence if $\Sigma A = \Sigma B$ then $A = B$, without restriction on A and B.*

Proof. By induction on $\min(r, k)$. The result is immediate if $r = 0$. Triangulate both polyhedra so that we have a simplicial homeomorphism $\Sigma^r X \overset{h}{\cong} \Sigma^k Y$. Each triangulation is assumed to be a subdivided join triangulation.

Suppose $r \leqslant k$. Let ξ be a point of Σ^r and put $x = h\xi$. Then $\mathrm{lk}\, x = \mathrm{lk}\, \xi = \Sigma^{r-1} X$. If x lies in the factor Σ^k then $\mathrm{lk}\, x = \Sigma^{k-1} Y$, and the result now follows by induction.

Otherwise, by (lk 4), $\mathrm{lk}\, x = \Sigma^k(y, Y)$ for some $y \in Y$, whether x lies in Y or not. Write $\mathrm{lk}(y, Y) = \Sigma^s Z$ where s is maximal. Then $\Sigma^{r-1} X = \Sigma^{k+s} Z$, and by induction we have $r - 1 = k + s$. But $r \leqslant k$ so this is impossible.

The fact that a ball can be taken as the join either of two lower dimensional balls, or of a ball and a sphere means that we cannot always cancel a factor T. The best result is the following.

THEOREM 2. *Maximal balls cancel. Suppose $T^r X = T^k Y$ where X and Y are not cones or suspensions. Then $k = r$ and $X = Y$.*

Proof. Suppose $r \leqslant k$, and write $T^r = \Sigma^{r-1} T$. Then $\Sigma^{r-1} TX = \Sigma^{r-1} T^m Y$ and so, by Theorem 1, $TX = T^m Y$ with $m \geqslant 1$. Triangulate so that $TX \overset{h}{\cong} T^m Y$, write ξ for the point T and $x = h\xi$.

If x lies in Y or between T^m and Y then its link is a cone or a suspension, which is impossible by the choice of $X = \mathrm{lk}\, \xi$. Hence x lies in T^m. Now $\mathrm{lk}\, x = \Sigma^{m-1} Y$ or $T^{m-1} Y$ and so we must have $m = 1$ and $X = Y$.

Definition. A *reduced* polyhedron is one which is not a cone or a suspension.

COROLLARY TO THEOREMS 1 AND 2. *Any polyhedron factors uniquely as a ball or sphere joined to a reduced polyhedron.*

Proof. There are three cases to consider, (i) $\Sigma^r X = \Sigma^k Y$, (ii) $T^r X = T^k Y$ and (iii) $\Sigma^r X = T^k Y$, where X and Y are reduced.

Theorems 1 and 2 give uniqueness in the first two cases. It remains to show that (iii) is impossible.

Take the cone on each side. Then $T\Sigma^r X = T^{r+1} X = T^{k+1} Y$, and hence $r = k$ and $X = Y$. Thus $\Sigma^r X = T^r X = \Sigma^{r-1} TX$, and, we can cancel the factor Σ^{r-1} to give $\Sigma X = TX$. Now this is impossible, for there are two points in ΣX which have a reduced link, while there is only one such point in TX.

Two further results about links which follow from Theorem 1 can now be given.

(lk 5) Let x be a simplex of X, and x' a simplex in some subdivision such that the interior of x' lies in the interior of x. Then $\mathrm{lk}\, x' = \Sigma^k \mathrm{lk}\, x$, where k is the codimension of x' in x.

Proof. Let y be a point in the interior of x', and let $n = \dim x'$. Then $\mathrm{lk}\, y = \Sigma^n \mathrm{lk}\, x' = \Sigma^{n+k} \mathrm{lk}\, x$, by lk 3. We now cancel Σ^n to give the result.

(lk 6) *If x is a simplex in some subdivision of AB, then there are simplexes a, b in the chosen join triangulation such that the interior of x lies in the interior of ab.* From (lk 2) and (5) we have $\mathrm{lk}\, x = \Sigma^k \mathrm{lk}(a, A)\, \mathrm{lk}(b, B)$, where k is the codimension of x in ab.

LEMMA 1. *Suppose $AB \overset{h}{\cong} XY$, where A is reduced, and suppose that $b = h\beta$ lies between X and Y, where β is a principal simplex of B. Then there are non-empty simplexes x and y of the initial triangulations of X and Y such that $A = \mathrm{lk}(x, X)\,\mathrm{lk}(y, Y)$. Further, there is a simplex β' of B whose image $h\beta'$ lies in X, such that $A\,\mathrm{lk}(\beta', B) = \mathrm{lk}(x, X)\,Y$.*

Proof. By (lk 6), b determines x and y with $A = \mathrm{lk}\,\beta = \mathrm{lk}\,b = \Sigma^r\,\mathrm{lk}(xy)$ where r is the codimension of b in xy. Since A is reduced, we have $r = 0$ and the first part of the result.

In the subdivision of XY the simplex xy is divided into a finite number of simplexes of codimension 0. Let b_1 and b_2 be two such simplexes which have a common top dimensional face c, and let a_1 and a_2 be their respective vertices opposite c. Suppose that b_1 is the image of a principal simplex β_1 in B. Then $A = \mathrm{lk}\,\beta_1 = \mathrm{lk}\,b_1 = \mathrm{lk}(xy) = \mathrm{lk}\,b_2$. Also c lies in the interior of xy, so by (lk 5) $\mathrm{lk}\,c = \Sigma A$.

The two suspension points in $\mathrm{lk}\,c$, regarded as an actual subcomplex of XY, must be a_1 and a_2. For a_1 and a_2 both lie in $\mathrm{lk}\,c$, and their link in $\mathrm{lk}\,c$ is A, which is reduced, while the suspension points in ΣA are the only points with reduced link.

Write $\gamma = h^{-1}c$, and then $\mathrm{lk}\,\gamma = \mathrm{lk}(\gamma, B)A$ so $\Sigma A = \mathrm{lk}(\gamma, B)A$. From the dimension, $\mathrm{lk}(\gamma, B)$ consists of a finite number of points, and since each has reduced link we must have $\mathrm{lk}(\gamma, B) = \Sigma$. Write α_1 and α_2 for the actual points in $\mathrm{lk}(\gamma, B)$. Then the link of α_1 in $\mathrm{lk}\,\gamma$ is the link of the simplex $\alpha_1\gamma$ of B in the whole complex. This simplex has empty link in B, hence its link is A. Thus the image of α_1 must be a suspension point in $\mathrm{lk}\,c$, that is $h\alpha_1$ and $h\alpha_2$ must be a_1 and a_2. Hence $\beta_2 = h^{-1}b_2$ lies entirely in B and is principal.

Now any two top-dimensional simplexes of xy can be joined by a chain of simplexes such as b_1 and b_2, and one such simplex, b, comes from B. Hence the whole of xy comes from B. In particular one of these simplexes must have a face b' lying in x with the same dimension as x. If we take $\beta' = h^{-1}b'$, then

$$A\,\mathrm{lk}(\beta', B) = \mathrm{lk}\,\beta' = \mathrm{lk}\,b' = \mathrm{lk}\,x = \mathrm{lk}(x, X)\,Y.$$

LEMMA 2. *Any indecomposable polyhedron except T or Σ is prime.*

Proof. Suppose A is indecomposable and $AB \overset{h}{\cong} XY$. Let β be a principal simplex of B and let $b = h\beta$. If b lies in one of the factors, say X, then, by (lk 2) and (6), $A = \mathrm{lk}\,b = \mathrm{lk}(b, X)\,Y$. Now A is indecomposable, so $\mathrm{lk}(b, X) = \varnothing$ and $A = Y$. Otherwise, by Lemma 1, we can write $A = \mathrm{lk}(x, X)\,\mathrm{lk}(y, Y)$. One of these factors, say $\mathrm{lk}(x, X)$, must be empty. Now also by Lemma 1 there is a simplex β' of B such that

$$A\,\mathrm{lk}(\beta', B) = \mathrm{lk}(x, X)\,Y.$$

Since $\mathrm{lk}(x, X) = \varnothing$ we have shown that A divides Y.

LEMMA 3. *The join of two reduced polyhedra is also reduced.*

Proof. By induction on the dimension of the join.

Suppose X and Y are reduced but XY is not. We can then write $XY \overset{h}{\cong} ZB$ where $Z = T^r$ or Σ^r and B is reduced. Let ξ be a principal simplex of X. Then $h\xi$ cannot lie in B, since $Y = \mathrm{lk}\,\xi$ is reduced. If it lies between B and Z then, by Lemma 1, there exists a non-empty simplex ξ' of X with $h\xi'$ in Z. Similarly there exists a non-empty η' of Y whose image lies in Z. Now the factors X and Y are disjoint in XY, and both contain simplexes mapping to Z, so Z must be disconnected, i.e. $Z = \Sigma$.

Suppose $X = X_1X_2$, then the same argument applied to X_1 and X_2Y, which is reduced by induction, shows that X_1 and similarly X_2 as well as Y must contain simplexes mapping to Σ, which is impossible. There only remains the case $XY \overset{h}{\cong} \Sigma B$ where X and Y are indecomposable. Write B as a product of indecomposable factors. These factors, and also X and Y, are prime by Lemma 2. This fact, and consideration of the dimension shows that $X = Y = B$ and hence dim $X = 0$. Now X has at least 3 points, so there are more than two points in X^2 with reduced link, but only two in ΣX.

THEOREM 3. *The semigroup of polyhedra is the direct product of the semigroup of balls and spheres and the semigroup of reduced polyhedra.*

Proof. This follows at once from Lemma 3 and the Corollary to Theorems 1 and 2.

THEOREM 4. *The decomposition of any reduced polyhedron into indecomposable factors is unique up to order.*

Proof. Since the indecomposable factors which occur must be prime, it is enough to show that if $AB = AC$ then $B = C$, where A, B and C are reduced. We shall proceed by induction on dim AB, which starts at dimension 0, for then one of the factors will be empty. We shall prove, using the induction hypothesis in dimensions less than dim X^rY, that if $X^rY = X^rZ$, where all polyhedra are reduced and X is any indecomposable which does not divide Y, then $Y = Z$. The induction hypothesis in dimension dim X^rY will then follow easily.

Suppose $X^rY \overset{h}{\cong} X^rZ$, where $X^r \neq \varnothing$, and let ξ be a principal simplex of X^r and $x = h\xi$. If x lies in X^r then Z divides lk $x = lk\ \xi = Y$, and so $Z = Y$ by dimension. It is not possible for x to lie in Z, otherwise X^r divides Y. Suppose x lies between X^r and Z; then by Lemma 1 there are non-empty simplexes $\bar{x}$, z and ξ' with $Y = \text{lk}\ (\bar{x}, X^r)\ \text{lk}(z, Z)$ and $\text{lk}(\xi', X^r)\ Y = X^r\ \text{lk}(z, Z)$. All the polyhedra mentioned must be reduced, since they either divide Y or the join of two reduced polyhedra, and since z is non-empty we have

$$\dim \text{lk}\ z = \dim X^r\,\text{lk}(z, Z) < \dim X^rZ.$$

Now $\text{lk}(\xi', X^r)\,\text{lk}(\bar{x}, X^r)\,\text{lk}(z, Z) = X^r\,\text{lk}(z, Z)$, so by the induction hypothesis $\text{lk}(\xi', X^r)\ \text{lk}(\bar{x}, X^r) = X^r$. Each factor on the left must be a power of X, since any indecomposable factor of either must divide X. Now $\dim \text{lk}(\xi, X^r) < \dim X^r$ so $\text{lk}(\bar{x}, X^r) = X^s$ for $s > 0$, which is impossible since it divides Y.

COROLLARY. *For any polyhedra A, B and C, if $AB = AC$ then either* (i) $B = C$ or (ii) $B = TX$, $C = \Sigma X$ *and* $A = TY$ *for some X and Y.*

Proof. A, B and C split up as a sphere or ball joined to the reduced polyhedra A', B' and C', and $A'B' = A'C'$ by Theorem 3. Then $B' = C'$ by Theorem 4. Hence, either $B = C$, or $B = T^rB'$ and $C = \Sigma^rB'$. Then A must be T^sA' for $s > 0$ and so we have (ii) with $X = T^{r-1}B'$ and $Y = T^{s-1}A'$.

§4. SELF-HOMEOMORPHISMS OF POLYHEDRA

We have shown something of the rigidity of structure of a polyhedron which is the join of a number of factors. We can now say precisely what the images of the various factors of a reduced polyhedron X are under any PL homeomorphism of X to itself.

THEOREM 5. *If X is the join $*_i X_i^{r_i}$ of reduced indecomposable factors X_i, with $X_i \neq X_j$ for $i \neq j$, and h is a PL homeomorphism from X to itself, then h sends each factor $X_i^{r_i}$ to itself, and $h \mid X_i^{r_i}$ permutes the factors X_i.*

Proof. We have shown in the proof of Lemma 4 that h does send each factor $X_i^{r_i}$ to itself. It remains to prove that if $Y = A^r$, with A reduced and indecomposable, than any homeomorphism h of Y to itself permutes the factors A.

Suppose $A^r \overset{h}{\cong} A^r$. For any $s < r$ write the lefthand side as $A^s \,.\, A^{r-s}$. Then we show that the image of any simplex of this factor A^s under h lies in some other factor A^s.

Let α be a principal simplex of A^s. Then $h\alpha$ is contained in the join of simplexes $a_1, \ldots, d_r$ of a join triangulation, one from each factor of A^r, where some of the a_i may be empty. Since A is reduced we can assume that the a_i are chosen so that

$$A^{r-s} = \mathrm{lk}\,\alpha = \mathrm{lk}(a_1, A)\,.\,\mathrm{lk}(a_2, A) \cdots \mathrm{lk}(a_r, A).$$

Now by the unique factorisation of reduced polyhedra exactly s of these factors are empty, and the others, suppose the last $r - s$ are A. Then for $i > s$, $\mathrm{lk}(a_1, A) = A$ so $a_i = \varnothing$. Thus $h\alpha$ lies in the join of the first s factors A.

So for each s the sets $\cup A^s$ are invariant under h. In particular, the set $\cup A$ is invariant under h, so if A is connected, then h must permute the disjoint factors A.

Call the factors $A_1, \ldots, A_r$, and suppose that A is not connected. Suppose that there are two points α_1 and α_2 in different components of one of the factors, A_1 say, such that $a_1 = h\alpha_1$ and $a_2 = h\alpha_2$ lie in different factors A_k and A_l. Consider the image of the line $a_1 a_2 \subset \cup A^2$ under h^{-1}. No points of this image other than α_1 and α_2 lie in $\cup A$. Now any path joining two points x and y in $\cup A^2$ where $x \in A_i A_j$ and $y \in A_k A_l$ with $\{i, j\} \neq \{k, l\}$ must contain points of $\cup\, A$. Thus the path $h^{-1} a_1 a_2$ must lie entirely in $A_1 A_j$ for some j, with no points lying in A_j. Such a path will project to A_1 to give a path joining α_1 to α_2.

THEOREM 6. *In the join X above, h sends any subjoin $\bar{X} = \underset{i \in I}{*} X_i$ into the subjoin $\underset{i \in I}{*} (hX_i)$.*

Proof. Let A' denote the complementary factor of A in the join X. Then for an indecomposable factor X_i, $h(X_i') = (hX_i)'$. For let x be a principal simplex of X_i'. Then since X_i and X_i' are disjoint, hx cannot lie in hX_i, nor using Lemma 1 can it lie between hX_i and its complement. So $hx \in (hX_i)'$.

Now the join $\bar{X}$ can be written as $\bigcap_{j \notin I} X_j'$. Then

$$h\bar{X} = \bigcap_{j \notin I} h(X_i') = \bigcap_{j \notin I} (hX_i)' = \underset{i \in I}{*} (hX_i).$$

§5. A REMARK ON THE PRODUCT OF POLYHEDRA

Define the *intrinsic skeleton* X_i of polyhedron X as the intersection of the i-skeletons of all triangulations of X, and the *intrinsic dimension* of a point x in X to be k when $\mathrm{lk}(x, X) = \Sigma^k A$, with k maximal. Armstrong [1] shows that the intrinsic skeleton X_i consists of all points with intrinsic dimension $\leq i$.

Call the polyhedron A the *intrinsic link* of the point x. This intrinsic link may contain at most one factor T. By analogy with manifolds we shall call x a *boundary point* if the intrinsic link of x is a cone. Then in a polyhedron without boundary points the intrinsic link of every point is reduced.

THEOREM 7. *Suppose X and Y are polyhedra without boundary. If $Z = X \times Y$, then* $Z_n = \bigcup_{i+j=n} X_i \times Y_j$.

Proof. Suppose $x \in X_i$ *and* $y \in Y_j$. Then the link of (x, y) in $Z = \mathrm{lk}(x, X)\,\mathrm{lk}(y, Y)$ (see [2]).

So $\mathrm{lk}(x, y) = \Sigma^i A \Sigma^j B = \Sigma^{i+j} AB$. But A and B are reduced, so by Lemma 3 AB is also reduced. Then the intrinsic dimension of (x, y) is $i+j$.

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